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Fractional-Order Systems and Fractional-Order Controllers

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Dynamic systems of an arbitrary real order (fractional-order systems) are considered. A concept of a fractional-order $PI^{\lambda}D^{\mu}$-controller, which involves fractional-order integrator and fractional-order differentiator, is proposed. A method for study of systems of an arbitrary real order is presented. The method is based on the Laplace transform formula for a new function of the Mittag-Leffler type. Explicit analytical expressions for the unit-step and unit-impulse response of a linear fractional-order system with fractional-order controller are given for the open and closed loop. An example demonstrating the use of the proposed method and the advantages of the proposed $PI^{\lambda}D^{\mu}$-controllers is given.

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Contents

Introduction

Recently some authors have considered systems described by fractional-order state equations (Bagley and Torvik (1984); Bagley and Calico (1991); Makroglou, Miller and Skaar (1994)), which means equations involving so-called fractional derivatives and integrals (e.g., Oldham and Spanier (1974)).

Those new models are more adequate than the previously used integer-order models. This was demonstrated, for instance, by Caputo (1969), Nonnenmacher and Glöckle (1991) and Fierdrich (1991). Important fundamental physical considerations in favour of the use of fractional-derivative based models were given by Caputo and Mainardi (1971) and Westerlund (1994). Fractional-order derivatives and integrals provide a powerful instrument for the description of memory and hereditary properties of different substances. This is the most significant advantage of the fractional-order models in comparison with integer-order models, in which, in fact, such effects are neglected.

However, because of the absense of appropriate mathematical methods, fractional-order dynamic systems were studied only marginally in the theory and practice of control systems. Some sucessful attempts were made by Oustaloup (1988); Axtell and Michael (1990); Bagley and Calico (1991); Kaloyanov and Dimitrova (1992); Makroglou, Miller and Skaar (1994), but the study in the time domain has been almost avoided.

In this paper some effective and easy-to-use tools for the time-domain analysis of fractional-order systems are presented. A concept of a $PI^{\lambda}D^{\mu}$-controller, involving fractional-order integrator and fractional-order differentiator, is introduced. An example is provided to demonstrate the necessity of such controllers for the more efficient control of fractional-order systems.

All computations were performed in MATLAB for Windows, version 4.0.

Chapter 1

Fractional-Order Systems and Fractional-Order Controllers

This paper is a natural continuation of our previous work (Podlubny, 1994), which we recommend to readers interested in system response to arbitrary input. However, in this paper we turn from purely mathematical aspects of the fractional calculus to application of the fractional calculus in the control theory.

1.1 Fractional-order control system

Let us consider a simple unity-feedback control system shown in Fig. 1.1, where $G(s)$ is the transfer function of the controlled system, $G_c(S)$ is the transfer of the controller, $W(s)$ is an input, $E(s)$ is an error, $U(s)$ is controller's output, $Y(s)$ is system's output.

Contrary to the traditional approach, we will consider *the transfer functions of an arbitrary real order*. We call such systems the fractional-order systems. They include, in particular, traditional integer-order systems.

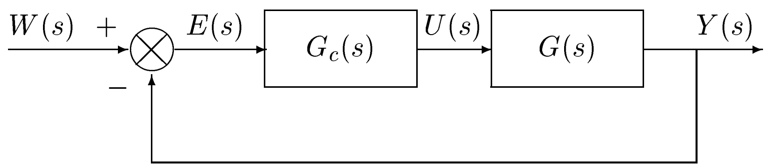

Figure 1.1: *Simple unity-feedback control system.*

1.2 Fractional-order transfer functions

Let us consider the fractional-order transfer function (FOTF) given by the following expression:

$$G_n(s) = \frac{1}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}} + \ldots + a_1 s^{\beta_1} + a_0 s^{\beta_0}}, \tag{1.1}$$

where β_k, $(k = 0, 1, \ldots, n)$ is an arbitrary real number,

$\beta_n > \beta_{n-1} > \ldots > \beta_1 > \beta_0 > 0$,

a_k, $(k = 0, 1, \ldots, n)$ is an arbitrary constant.

In the time domain, the FOTF (1.1) corresponds to n-term fractional-order differential equation (FDE)

$$a_n D^{\beta_n} y(t) + a_{n-1} D^{\beta_{n-1}} y(t) + \ldots + a_1 D^{\beta_1} y(t) + a_0 D^{\beta_0} y(t) = u(t) \tag{1.2}$$

where $D^\gamma \equiv {}_0D_t^\gamma$ is Caputo's fractional derivative of order γ with respect to variable t and with the starting point at $t = 0$ (Caputo (1967, 1969)):

$${}_0D_t^\gamma y(t) = \frac{1}{\Gamma(1-\delta)} \int_0^t \frac{y^{(m+1)}(\tau)d\tau}{(t-\tau)^\delta}, \qquad (\gamma = m + \delta, \quad m \in Z, \quad 0 < \delta \leq 1) \tag{1.3}$$

If $\gamma < 0$, then one has a fractional integral of order $-\gamma$:

$${}_0I_t^{-\gamma} y(t) = {}_0D_t^\gamma y(t) = \frac{1}{\Gamma(-\gamma)} \int_0^t \frac{y(\tau)d\tau}{(t-\tau)^{1+\gamma}}, \quad (\gamma < 0) \tag{1.4}$$

The Laplace transform of the fractional derivative defined by (1.3) is (Caputo (1967, 1969))

$$\int_0^\infty e^{-st} D^\gamma y(t) dt = s^\gamma Y(s) - \sum_{k=0}^{m} s^{\gamma-k-1} y^{(k)}(0) \tag{1.5}$$

For $\gamma < 0$ (i.e., for the case of a fractional integral) the sum on the right-hand side must be omitted.

It is worth mentioning here that from the pure mathematical point of view there are different ways to interpolate between integer-order multiple integrals and derivative. The most widely known and precisely studied is the Riemann-Liouville definition of fractional derivatives (e.g., Oldham and Spanier (1974); Samko, Kilbas and Maritchev (1987); Miller and Ross (1993)). The main advantage of Caputo's definition in comparison with the Riemann-Liouville definition is that it allows consideration of easily interpreted conventional initial conditions such as $y(0) = y_0, y'(0) = y_1$, etc. Moreover, Caputo's derivative of a constant is bounded (namely, equal to 0), while Riemann-Liouville derivative of a constant is unbounded at $t = 0$. The only exception is if one takes $t = -\infty$ as the starting point (lower limit) in the Riemann-Liouville definition. In such a case, the Riemann-Liouville fractional derivative of a constant is also 0, and this was used by Ochmann and Makarov (1993). However, one interested in transient processes could not accept placement of the starting point in $-\infty$, and in such cases Caputo's definition seems to be the most appropriate compared to others.

Formula (1.5) is a particular case of a more general formula given by Podlubny (1994) for the Laplace transform of a so-called sequential fractional derivative (Miller and Ross, 1993)

To find the unit-impulse and unit-step response of the fractional-order system described by FDE (1.2), we need to evaluate the inverse Laplace transform of the function $G_n(s)$.

The problem of the Laplace inversion of (1.1), however, can appear in any field of applied mathematics, physics, engineering etc., where the Laplace transform method is used. This fact along with the absense of the necessary inversion formula in tables and handbooks on the Laplace transform motivated us to give a general solution to this problem in the two following sections.

1.3 New function of the Mittag-Leffler type

The so-called Mittag-Leffler function in two parameters $E_{\alpha,\beta}(z)$ was introduced by Agarwal (1953). His definition was later modified by Erdélyi *et al.* (1955) to be

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \qquad (\alpha > 0, \quad \beta > 0) \tag{1.6}$$

Its k-th derivative is given by

$$E^{(k)}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{(j+k)!\; z^j}{j!\; \Gamma(\alpha j + \alpha k + \beta)}, \qquad (k = 0, 1, 2, ...) \tag{1.7}$$

We find it convenient to introduce the function

$$\mathcal{E}_k(t, y; \alpha, \beta) = t^{\alpha k + \beta - 1} E^{(k)}_{\alpha,\beta}(y t^{\alpha}), \quad (k = 0, 1, 2, \ldots) \tag{1.8}$$

Its Laplace transform was (in other notation) evaluated by Podlubny (1994):

$$\int_0^{\infty} e^{-st} \mathcal{E}_k(t, \pm y; \alpha, \beta) dt = \frac{k!\; s^{\alpha - \beta}}{(s^{\alpha} \mp y)^{k+1}}, \qquad (Re(s) > |y|^{1/\alpha}). \tag{1.9}$$

Another convenient property of $\mathcal{E}_k(t, y; \alpha, \beta)$, which we use in this paper, is its simple fractional differentiation (Podlubny, 1994):

$${}_0D_t^{\lambda} \mathcal{E}_k(t, y; \alpha, \beta) = \mathcal{E}_k(t, y; \alpha, \beta - \lambda), \quad (\lambda < \beta). \tag{1.10}$$

Other properties of function $\mathcal{E}_k(t, y; \alpha, \beta)$, such as special cases, asymptotic behavior etc., can be obtained from (1.6)–(1.8) and the known properties (Erdélyi *et al.*, 1955) of the Mittag-Leffler function $E_{\alpha,\beta}(z)$.

1.4 General formula

Relationship (1.9) allows us to evaluate the inverse Laplace transform of (1.1) as follows.

Let $\beta_n > \beta_{n-1} > \ldots > \beta_1 > \beta_0 > 0$. Then

$$
\begin{aligned}
G_n(s) &= \frac{1}{a_n s^{\beta_n} + a_{n-1}s^{\beta_{n-1}}} \frac{1}{1 + \dfrac{\sum\limits_{k=0}^{n-2} a_k s^{\beta_k}}{a_n s^{\beta_n} + a_{n-1}s^{\beta_{n-1}}}} \\
&= \frac{a_n^{-1} s^{-\beta_{n-1}}}{s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n}} \frac{1}{1 + \dfrac{a_n^{-1}s^{-\beta_{n-1}}\sum\limits_{k=0}^{n-2} a_k s^{\beta_k}}{s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n}}} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m a_n^{-1} s^{-\beta_{n-1}}}{\left(s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n}\right)^{m+1}} \left(\sum_{k=0}^{n-2} \left(\frac{a_k}{a_n}\right) s^{\beta_k - \beta_{n-1}} \right)^m \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m a_n^{-1} s^{-\beta_{n-1}}}{\left(s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n}\right)^{m+1}} \qquad \text{(continued)} \\
&\qquad \sum_{\substack{k_0+k_1+\ldots+k_{n-2}=m \\ k_0\geq 0;\ldots,k_{n-2}\geq 0}} (m; k_0, k_1, \ldots, k_{n-2}) \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n}\right)^{k_i} s^{(\beta_i-\beta_{n-1})k_i} \\
&= \frac{1}{a_n} \sum_{m=0}^{\infty} (-1)^m \sum_{\substack{k_0+k_1+\ldots+k_{n-2}=m \\ k_0\geq 0;\ldots,k_{n-2}\geq 0}} (m; k_0, k_1, \ldots, k_{n-2}) \qquad \text{(continued)} \\
&\qquad \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n}\right)^{k_i} \frac{s^{-\beta_{n-1}+\sum\limits_{i=0}^{n-2}(\beta_i-\beta_{n-1})k_i}}{\left(s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n}\right)^{m+1}}
\end{aligned} \tag{1.11}
$$

where $(m; k_0, k_1, \ldots, k_{n-2})$ are the multinomial coefficients (Abramowitz and Stegun, 1964).

The term-by-term inversion, based on the general expansion theorem for the Laplace transform given in (Doetsch, 1956), using (1.9) gives the final expression for the inverse Laplace transform of function $G_n(s)$:

$$
\begin{aligned}
g_n(t) &= \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0+k_1+\ldots+k_{n-2}=m \\ k_0\geq 0;\ldots,k_{n-2}\geq 0}} (m; k_0, k_1, \ldots, k_{n-2}) \qquad \text{(continued)} \\
&\quad \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n}\right)^{k_i} \mathcal{E}_m(t, -\frac{a_{n-1}}{a_n}; \beta_n - \beta_{n-1}, \beta_n + \sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j)
\end{aligned} \tag{1.12}
$$

Further inverse Laplace transforms can be obtained by combining (1.10) and (1.12). For instance, let us take

$$F(s) = \sum_{i=1}^{N} b_i s^{\alpha_i} G_n(s), \tag{1.13}$$

where $\alpha_i < \beta_n$, $(i = 1, 2, \ldots, N)$. Then the inverse Laplace transform of $F(s)$ is

$$f(t) = \sum_{i=1}^{N} b_i D^{\alpha_i} g_n(t), \tag{1.14}$$

where the fractional derivatives of $g_n(t)$ are evaluated with the help of (1.10).

1.5 The unit-impulse and unit-step response

The unit-impulse response of the fractional-order system with the transfer function (1.1) is given by formula (1.12), i.e. $y_{impulse}(t) = g_n(t)$.

To find the unit-step response $y_{step}(t)$, one has to integrate (1.12) with the help of (1.10). The result is:

$$y_{step}(t) = \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0+k_1+\ldots+k_{n-2}=m \\ k_0 \geq 0; \ldots, k_{n-2} \geq 0}} (m; k_0, k_1, \ldots, k_{n-2}) \quad \text{(continued)}$$

$$\prod_{i=0}^{n-2} \left(\frac{a_i}{a_n} \right)^{k_i} \mathcal{E}_m(t, -\frac{a_{n-1}}{a_n}; \beta_n - \beta_{n-1}, \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j + 1) \tag{1.15}$$

1.6 Some special cases

For the illustration, we give three following particular cases of (1.12) and (1.15).

1)

$$G_2(s) = \frac{1}{as^\alpha + b}, \qquad (\alpha > 0)$$

$$\left. \begin{array}{r} y_{impulse}(t) = g_2(t) \\ y_{step}(t) = {}_0D_t^{-1} g_2(t) \end{array} \right\} = \frac{1}{a} \mathcal{E}_0(t, -\frac{b}{a}; \alpha, \alpha + \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\}) \tag{1.16}$$

2)

$$G_3(s) = \frac{1}{as^\beta + bs^\alpha + c}, \qquad (\beta > \alpha > 0)$$

$$\left. \begin{array}{r} y_{impulse}(t) = g_3(t) \\ y_{step}(t) = {}_0D_t^{-1} g_3(t) \end{array} \right\} = \frac{1}{a} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{c}{a} \right)^k \mathcal{E}_k(t, -\frac{b}{a}; \beta - \alpha, \beta + \alpha k + \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\}) \tag{1.17}$$

3)

$$G_4(s) = \frac{1}{as^\gamma + bs^\beta + cs^\alpha + d}, \qquad (\gamma > \beta > \alpha > 0)$$

$$\left.\begin{array}{r} y_{impulse}(t) = g_4(t) \\ y_{step}(t) = {}_0D_t^{-1}\, g_4(t) \end{array}\right\} =$$

$$= \frac{1}{a}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{-d}{a}\right)^m \sum_{k=0}^{m}\binom{m}{k}\left(\frac{c}{d}\right)^k \mathcal{E}_m(t, -\frac{b}{a}; \gamma-\beta, \gamma+\beta m-\alpha k+\left\{\begin{array}{c} 0 \\ 1 \end{array}\right\}) \tag{1.18}$$

Integrating the unit-step response with the help of (1.10), we obtain the unit-ramp response. Double integration of the unit-step response gives the response for the parabolic input. All those standard test input signals are frequently used in the control theory, and the above formulas provide explicit analytical expressions for the corresponding system responses.

1.7 $PI^\lambda D^\mu$-controller

As will be shown on an example below, a suitable way to the more efficient control of fractional-order systems is to use fractional-order controllers. We propose a generalization of the PID-controller, which can be called the $PI^\lambda D^\mu$-controller because of involving an integrator of order λ and differentiator of order μ. The transfer function of such a controller has the form:

$$G_c(s) = \frac{U(s)}{E(s)} = K_P + K_I s^{-\lambda} + K_D s^\mu, \qquad (\lambda,\, \mu > 0) \tag{1.19}$$

The equation for the $PI^\lambda D^\mu$-controller's output in the time domain is:

$$u(t) = K_P e(t) + K_I D^{-\lambda} e(t) + K_D D^\mu e(t) \tag{1.20}$$

Taking $\lambda = 1$ and $\mu = 1$, we obtain a classic PID-controller. $\lambda = 1$ and $\mu = 0$ give a PI-controller. $\lambda = 0$ and $\mu = 1$ give a PD-controller. $\lambda = 0$ and $\mu = 0$ give an gain.

All these classical types of PID-controllers are the particular cases of the fractional $PI^\lambda D^\mu$-controller (1.19). However, the $PI^\lambda D^\mu$-controller is more flexible and gives an opportunity to better adjust the dynamical properties of a fractional-order control system.

1.8 Open-loop system response

Let us delete the feedback in Fig. 1.1 and consider the obtained open loop with the $PI^\lambda D^\mu$-controller (1.19) and the fractional-order controlled system with the transfer function $G_n(s)$ given by expression (1.1).

In the time domain, this open-loop system is described by the fractional-order differential equation

$$\sum_{k=0}^{n} a_k D^{\beta_k} y(t) = K_P w(t) + K_I D^{-\lambda} w(t) + K_D D^\mu w(t) \tag{1.21}$$

The transfer function of the considered open-loop system is

$$G_{open}(s) = \left(K_P + K_I s^{-\lambda} + K_D s^{\mu}\right) G_n(s) \tag{1.22}$$

Since (1.22) has the same structure as (1.13), the inverse Laplace transform for $G_{open}(s)$ can be found with the help of formula (1.14). Therefore, the unit-step response of the considered fractional-order open-loop system is

$$g_{open}(t) = K_P g_n(t) + K_I D^{-\lambda} g_n(t) + K_D D^{\mu} g_n(t), \tag{1.23}$$

where $g_n(t)$ is given by (1.12).

To find the unit-step response, one should integrate (1.23) using formula (1.10).

1.9 Closed-loop system response

To obtain the unit-impulse and unit-step response for a closed-loop control system (Fig.1.1) with the $PI^{\lambda}D^{\mu}$-controller and the fractional-order controlled system with the transfer function $G_n(s)$ given by expression (1.1), one needs, at first, to replace $w(t)$ with $e(t) = w(t) - y(t)$ in equation (1.21). This step results in

$$\begin{aligned} \sum_{k=0}^{n} a_k D^{\beta_k} y(t) + K_P y(t) + K_I D^{-\lambda} y(t) + K_D D^{\mu} y(t) &= \\ = K_P w(t) + K_I D^{-\lambda} w(t) + K_D D^{\mu} w(t) & \end{aligned} \tag{1.24}$$

From (1.24) one obtains the following expression for the transfer function of the considered closed-loop system:

$$G_{closed}(s) = \frac{K_P s^{\lambda} + K_I + k_D s^{\mu+\lambda}}{\sum\limits_{k=0}^{n} a_k s^{\beta_k+\lambda} + K_P s^{\lambda} + K_I + K_D s^{\mu+\lambda}} \tag{1.25}$$

The unit-impulse response $g_{closed}(t)$ is then obtained by the Laplace inversion of (1.25), which could be performed by rearranging in decreasing order of differentiation the addends in the denominator of (1.25) and applying after that relationships (1.12) and (1.14). To find the unit-step response, one should integrate obtained unit-impulse response with the help of (1.10).

Chapter 2

Example

In this chapter we give an example showing the usefulness of the $PI^{\lambda}D^{\mu}$-controllers in comparison with conventional PID-controllers. We consider a fractional-order system, which plays the role of "reality", and its integer-order approximation, which plays the role of a "model". We show that the model fits the "reality" well.

However, the conventional PD-controller, designed on the base of the model, is shown to be not so suitable for the control of the fractional-order "reality"as one might expect.

A suitable way to the improvement of the control is to use a controller of a similar "nature" as the "reality", i.e. a fractional-order PD^{μ}-controller.

2.1 Fractional-order controlled system

Let us consider a fractional-order controlled system with the transfer function

$$G(s) = \frac{1}{a_2 s^{\beta} + a_1 s^{\alpha} + a_0} \tag{2.1}$$

where we take $a_2 = 0.8$, $a_1 = 0.5$, $a_0 = 1$, $\beta = 2.2$, $\alpha = 0.9$.

The fractional-order transfer function (2.1) corresponds in the time domain to the three-term fractional-order differential equation

$$a_2 y^{(\beta)}(t) + a_1 y^{(\alpha)}(t) + a_0 y(t) = u(t) \tag{2.2}$$

with zero initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$.

The unit-step response is found by (1.17):

$$y(t) = \frac{1}{a_2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{a_0}{a_2}\right)^k \mathcal{E}_k(t, -\frac{a_1}{a_2}; \beta - \alpha, \beta + \alpha k + 1) \tag{2.3}$$

2.2 Integer-order approximation

For comparison purposes, let us approximate the considered fractional-order system by a second-order system. Noticing that $\beta = 2.2$ and $\alpha = 0.9$ are close

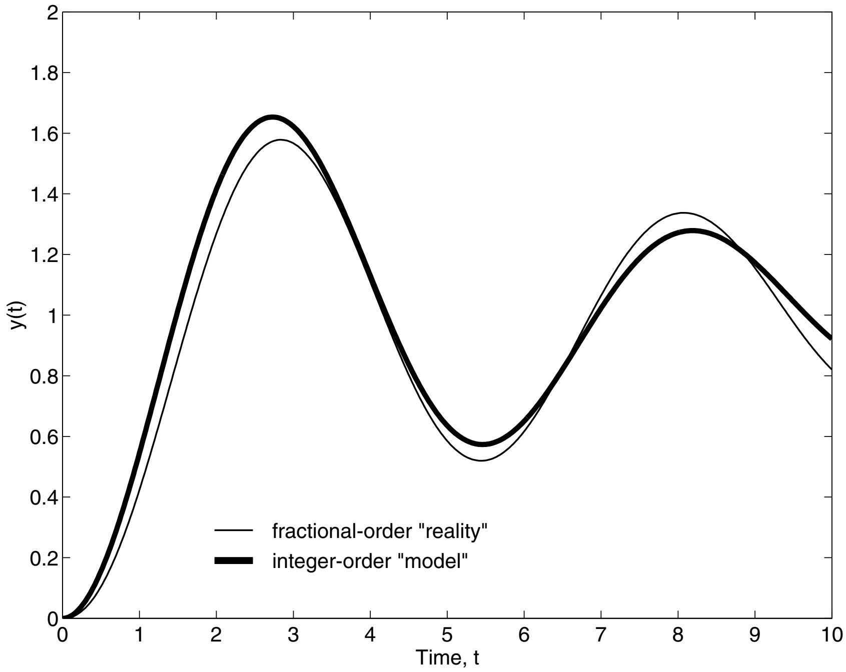

Figure 2.1: *Comparison of the unit-step response of the fractional-order system (thin line) and its approximation (thick line).*

to 2 and 1, respectively, one may expect good approximation. Using the least-squares method for the determination of coefficients of the resulting equation, we obtained the following approximating equation corresponding to (2.2):

$$\tilde{a}_2 y''(t) + \tilde{a}_1 y'(t) + \tilde{a}_0 y(t) = u(t) \tag{2.4}$$

with $\tilde{a}_2 = 0.7414$, $\tilde{a}_1 = 0.2313$, $\tilde{a}_0 = 1$.

The comparison of the unit-step response of systems described by (2.2) (original system) and (2.4) (approximating system) is shown in (2.1). The agreement seems to be satisfactory to build up the control strategy on the description of the original fractional-order system by its approximation.

2.3 Integer-order PD-controller

Since the above comparison of the unit-step responses shows good agreement, one may try to control the original system (2.2) by a controller designed for its approximation (2.4). This approach is, in fact, frequently used in practice, when one controls the real object by a controller designed for the model of that object.

The PD-controller with the transfer function

$$\tilde{G}_c(s) = \tilde{K} + \tilde{T}_d s \tag{2.5}$$

was designed so that a unit step signal at the input of the closed-loop system in Fig.1.1 will induce at the output an oscillatory unit-step response with stability measure $St = 2$ (this is equivivalent to the requirement that the system must settle within 5% of the unit step at the input in 2 seconds: $T_s \leq 2\mathrm{s}$) and damping ratio $\xi = 0.4$. In such a case, the coefficients for (2.5) take on the values $\tilde{K} = 20.5$ and $\tilde{T}_d = 2.7343$.

For comparison purposes, we also computed the integral of the absolute error (IAE)

$$I(t) = \int_0^t |e(t)| dt$$

for $t = 5$ s: $I(5) = 0.8522$.

Let us now apply this controller, designed for the optimal control of the approximating integer-order system (2.4), to the control of the approximated fractional-order system (2.2).

The differential equation of the closed loop with the fractional-order system defined by (2.1) and the integer-order controller defined by (2.5) has the following form:

$$a_2 y^{(\beta)}(t) + \tilde{T}_d y'(t) + a_1 y^{(\alpha)}(t) + (a_0 + \tilde{K}) y(t) = \tilde{K} w(t) + \tilde{T}_d w'(t) \qquad (2.6)$$

This is the four-term fractional differential equation, and the unit-step response of this system is found with the help of (1.18):

$$\begin{aligned} y(t) \;\; = \;\; & \frac{1}{a_2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{a_0 + \tilde{K}}{a_2} \right)^m \qquad \text{(continued)} \\ & \sum_{k=0}^{m} \binom{m}{k} \left(\frac{a_1}{a_0 + \tilde{K}} \right)^m \left\{ \tilde{K} \mathcal{E}_m(t, -\frac{\tilde{T}_d}{a_2}; \beta - 1, \beta + m - \alpha k + 1) + \right. \\ & \left. + \tilde{T}_d \mathcal{E}_m(t, -\frac{\tilde{T}_d}{a_2}; \beta - 1, \beta + m - \alpha k) \right\} \qquad (2.7) \end{aligned}$$

A comparison of the unit-step response of the closed-loop integer-order (approximating) system and the closed-loop fractional-order (approximated) system with the same integer-order controller, optimally designed for the approximating system, is shown in (2.2).

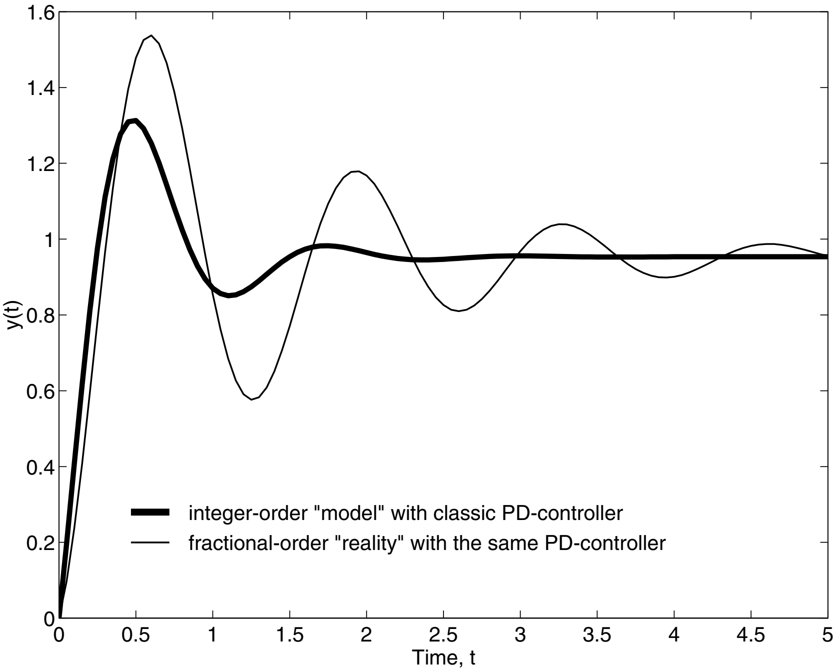

Figure 2.2: *Comparison of the unit-step response of the closed-loop integer-order system (thick line) and the closed-loop fractional-order system (thin line) with the same integer-order controller, optimally designed for the approximating integer-order system.*

One can see that the dynamic properties of the closed loop with the fractional-order controlled system and the integer-order controller, which was designed for the integer-order approximation of the fractional-order system, are considerably worse than the dynamic properties of the closed loop with the approximating integer-order system. The systems stabilizes slower and has larger surplus oscillations. Computations show that, in comparison with the integer-order "model", in this case the IAE within 5 s time interval is larger by 76%. Moreover, the closed loop with the fractional-order controlled system is more sensitive to changes in controller parameters. For example, at the change of $\tilde{T}_d$ to value 1, the closed loop with the fractional-order system (the "reality") is already unstable, whereas the closed loop with the approximating integer-order system (the "model") still shows stability (Fig.2.3).

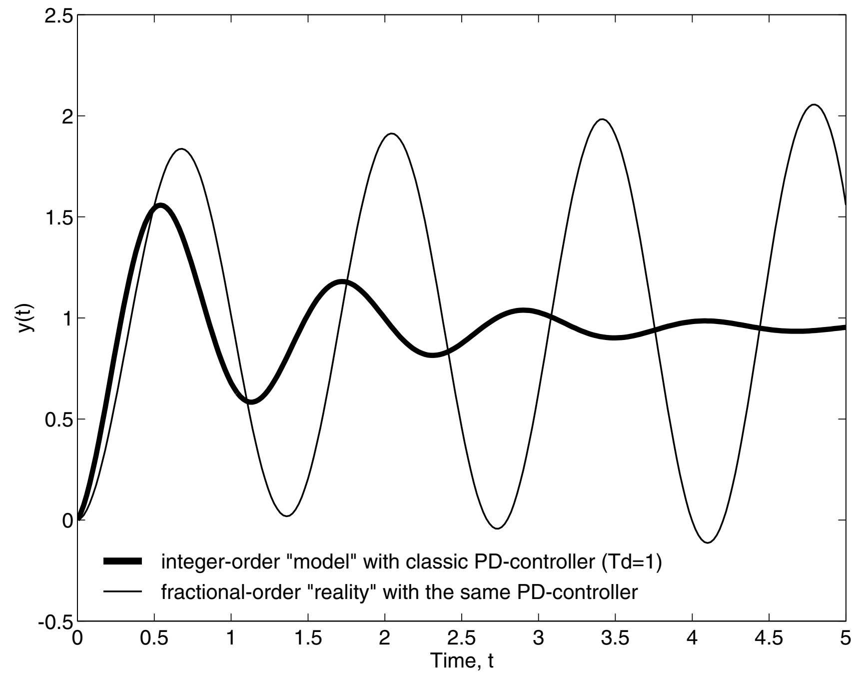

Figure 2.3: *Comparison of the unit-step response of the closed-loop integer-order system (thick line) and the closed-loop fractional-order system (thin line) with the same integer-order controller, optimally designed for the approximating system, for* $\tilde{T}_d = 1$.

2.4 Fractional-order controller

We see that disregarding the fractional order or the original system (2.2), replacing it with the approximating integer-order system (2.4) and application of the controller, designed for the approximating system, to the control of the original fractional-order system is not generally adequate.

An alternative and more successful approach in our example is to use the fractional-order PD^{μ}-controller characterized by the fractional-order transfer function

$$G_c(s) = K + T_d s^{\mu} \tag{2.8}$$

Let us take $\alpha < \mu < \beta$. The differential equation of the closed-loop control system with the fractional-order system transfer (2.1) and the fractional-order controller transfer (2.8) can be written in the form:

$$a_2 y^{(\beta)}(t) + T_d y^{(\mu)}(t) + a_1 y^{(\alpha)}(t) + (a_0 + K) y(t) = K w(t) + T_d w^{(\mu)}(t) \tag{2.9}$$

We are interested in the unit-step response of this system.

Using (1.18), (1.14) and (1.10), the following solution to equation (2.9) is

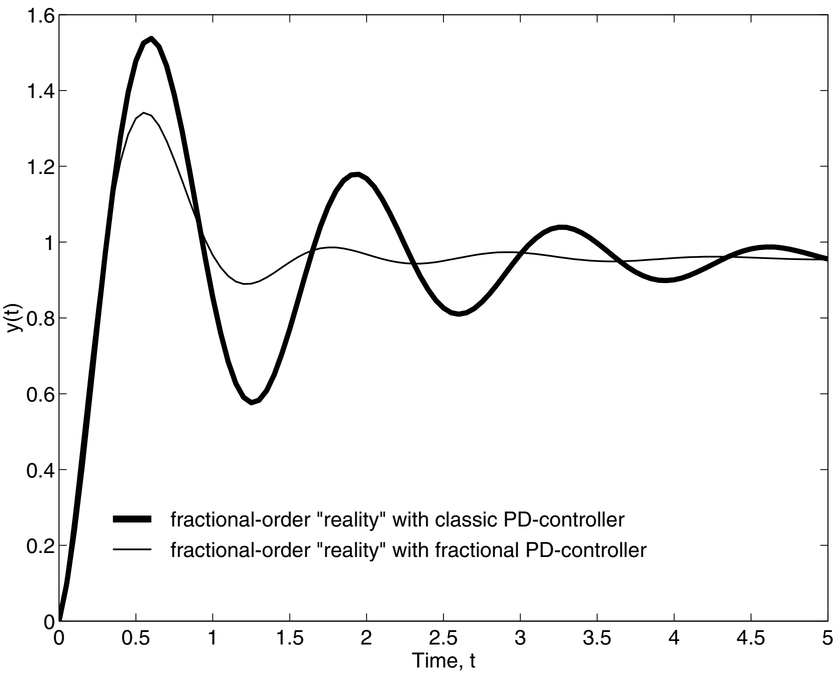

Figure 2.4: *Comparison of the unit-step response of the closed-loop fractional-order system with the conventional PD-controller controller, optimally designed for the approximating integer-order system (thick line), and with the PD^{μ}-controller (thin line).*

obtained:

$$
\begin{aligned}
y(t) \quad = \quad & \frac{1}{a_2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{a_0 + K}{a_2} \right)^m \qquad \text{(continued)} \\
& \sum_{k=0}^{m} \binom{m}{k} \left(\frac{a_1}{a_0 + K} \right)^k \Big\{ K\mathcal{E}_m(t, -\frac{T_d}{a_2}; \beta - \mu, \beta + \mu m - \alpha k + 1) + \\
& \qquad + T_d \mathcal{E}_m(t, -\frac{T_d}{a_2}; \beta - \mu, \beta + \mu m - \alpha k + 1 - \mu) \Big\} \qquad (2.10)
\end{aligned}
$$

In (2.4), the comparison of the unit-step response of the closed loop with the fractional-order system controlled by fractional-order PD^{μ}-controller with $K = \tilde{K}$, $T_d = 3.7343$ and $\mu = 1.15$ (the values of the parameters were found by computational experiments) and the unit-step response of the closed loop with the same system controlled by the integer-order PD-controller, designed for the approximating integer-order system, is given.

One can see that the use of the fractional-order controller leads to the improvement of the control of the fractional-order system.

Conclusion

We have shown that the proposed concept of the fractional-order $PI^{\lambda}D^{\mu}$-controller is a suitable way for the adequate control of the fractional-order systems.

Of course, for the physical realization of the $PI^{\lambda}D^{\mu}$-controller specific circuits are necessary: they must perform Caputo's fractional-order differentiation and integration. It should be mentioned that such fractional integrators and differentiators have already been described by Oldham and Spanier (1974); Oldham and Zoski (1983).

All the results of computations were also verified by the numerical solution of the initial-value problems for the corresponding fractional-order differential equations (Dorcak, Prokop and Kostial, 1994).

The most important limitation of the method presented in this paper is that only linear systems with constant coefficients can be treated. On the other hand, it allows consideration of a new class of dynamic systems (systems of an arbitrary real order) and new types of controllers.

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