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Fractional Calculus: Methods for Applications

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Topics to be discussed

- Interpretations of fractional-order operators
- Numerical fractional differentiation and numerical solution of FDEs
- Matrix approach: from CO to VO to DO
- Building fractional-order models: fitting using the Mittag-Leffler function
- “Least circles” method
- Related Matlab toolboxes
- Applications

Slovakia

Kosice and surroundings

Tatra Mountains

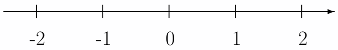
Caves

Is it in the middle of nowhere?

KOŠICE2013
EUROPEAN CAPITAL OF CULTURE

Warm-up quiz

. . . from integer to non-integer . . .



$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n}$$

$$x^n = e^{n \ln x}$$

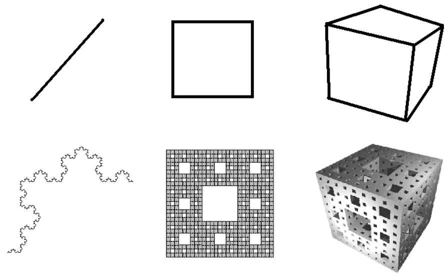
$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n,$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad x > 0,$$

$$\Gamma(n+1) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n!$$

. . . from integer to non-integer . . .

$D = 1$ $\quad D = 2$ $\quad D = 3$



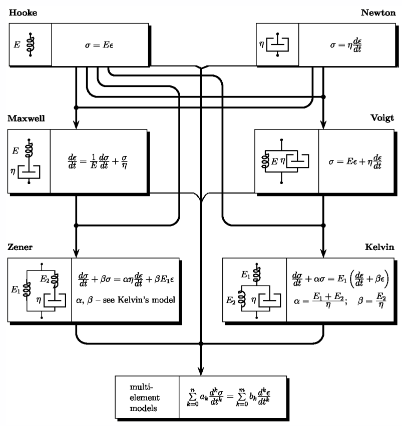
$D = 1.26$ $\quad D = 1.89$ $\quad D = 2.73$

Interpolation of operations

$$f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \frac{d^3 f}{dt^3}, \quad \ldots$$

$$f, \quad \int f(t) dt, \quad \int dt \int f(t) dt, \quad \int dt \int dt \int f(t) dt, \quad \ldots$$

$$\ldots, \quad \frac{d^{-2} f}{dt^{-2}}, \quad \frac{d^{-1} f}{dt^{-1}}, \quad f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \ldots$$



$$\sigma(t) + a_0 D_t^\alpha \sigma(t) = b\epsilon(t) + c_0 D_t^\alpha \epsilon(t)$$

“Fractional-order” physics?

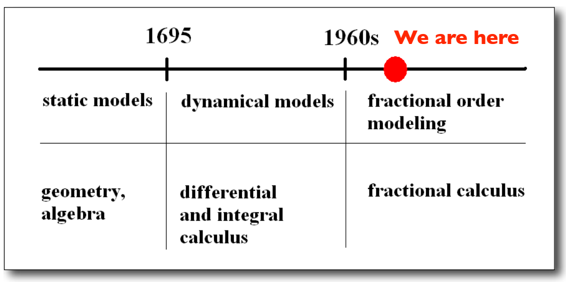
Hooke’s law: $F = kx$

Newton’s fluid: $F = kx'$ $\longrightarrow F(t) = k x^{(\alpha)}(t)$

Newton’s 2nd law: $F = kx''$

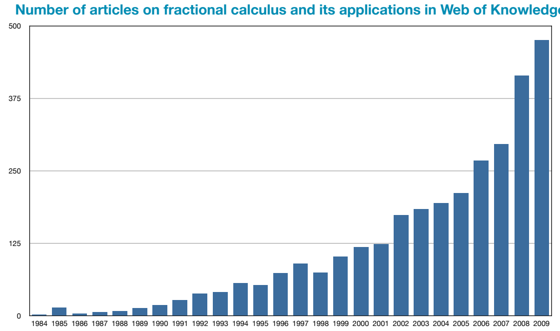
Diffusion–wave equation: $\dfrac{\partial^\alpha u}{\partial t^\alpha} = \dfrac{\partial^2 u}{\partial x^2}$

Fractional Calculus: a response to S&T needs



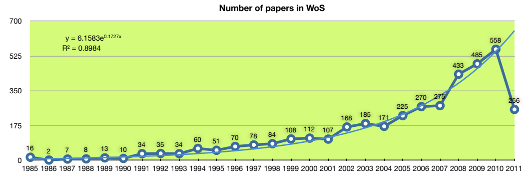
Rapid development and numerous applications

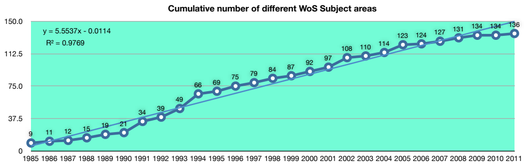
Number of articles on fractional calculus and its applications in Web of Knowledge



Zdroj: Web of Knowledge

Rapid development and numerous applications

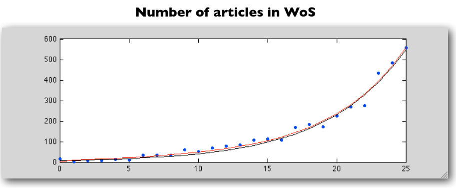




Rapid development and numerous applications

	Number of papers in WoS	Number of WoS Subject areas in particular year	Cumulative number of different WoS Subject areas
1985	16	9	9
1986	2	4	11
1987	7	3	12
1988	8	6	15
1989	13	8	19
1990	10	9	21
1991	34	24	34
1992	35	20	39
1993	34	30	49
1994	60	38	66
1995	51	36	69
1996	70	37	75
1997	78	30	79
1998	84	43	84
1999	108	36	87
2000	112	47	92
2001	107	46	97
2002	168	58	108
2003	185	53	110
2004	171	64	114
2005	225	66	123
2006	270	57	124
2007	275	58	127
2008	433	70	131
2009	485	76	134
2010	558	61	134
2011	256	53	136

Number of articles in WoS



Classical fitting by exponential

$y = 6.1583e^{0.1727x}$ $R2 = 1.0552$ e+04

Classical fitting by the Mittag-Leffler

$y = 4.2072\, E_{0.1559,\, 2.6082}(0.7987\, x^{0.1559})$

$R2 = 8.8504$ e+03

The current map of the fractional calculus

ALGERIA
ARGENTINA
AUSTRALIA
AUSTRIA
BELGIUM
BRAZIL
BULGARIA
BYELARUS
CANADA
CHILE
CROATIA
CUBA
CZECH REP
DENMARK
EGYPT
ENGLAND
ESTONIA
FINLAND
FRANCE
GERMANY
GREECE
GUADELOUPE
HUNGARY
INDIA
IRAN
IRELAND
ISRAEL
ITALY
JAPAN
JORDAN
KUWAIT
LEBANON
LITHUANIA
MACEDONIA
MALAYSIA
MEXICO
MOROCCO
NETHERLANDS
NEW ZEALAND
NIGERIA
NORTH IRELAND
NORWAY
PAKISTAN
P.R. CHINA
POLAND
PORTUGAL
QATAR
ROMANIA
RUSSIA
SAUDI ARABIA
SCOTLAND
SERBIA
MONTENEG
SINGAPORE
SLOVAKIA
SLOVENIA
SOUTH AFRICA
SOUTH KOREA
SPAIN
SWEDEN
SWITZERLAND
TAIWAN
THAILAND
TUNISIA
TURKEY
TURKMENISTAN
U ARAB EMIRATES
UKRAINE
USA
UZBEKISTAN
VENEZUELA
VIETNAM
WALES
YUGOSLAVIA

Data Source: Web of Knowledge

Fractional Calculus in WoK: 135 subject areas (applications)

ACOUSTICS
AGRICULTURAL ECONOMICS & POLICY
AGRICULTURAL ENGINEERING
AGRONOMY
ANESTHESIOLOGY
ASTRONOMY & ASTROPHYSICS
AUTOMATION & CONTROL SYSTEMS
BIOCHEMICAL RESEARCH METHODS
BIOCHEMISTRY & MOLECULAR BIOLOGY
BIOLOGY
BIOPHYSICS
BIOTECHNOLOGY & APPLIED MICROBIOLOGY
BUSINESS
BUSINESS, FINANCE
CARDIAC & CARDIOVASCULAR SYSTEMS
CELL BIOLOGY
CHEMISTRY, ANALYTICAL
CHEMISTRY, APPLIED
CHEMISTRY, INORGANIC & NUCLEAR
CHEMISTRY, MULTIDISCIPLINARY
CHEMISTRY, ORGANIC
CHEMISTRY, PHYSICAL
COMPUTER SCIENCE, ARTIFICIAL INTELLIGENCE
COMPUTER SCIENCE, CYBERNETICS
COMPUTER SCIENCE, HARDWARE & ARCHITECTURE
COMPUTER SCIENCE, INFORMATION SYSTEMS
COMPUTER SCIENCE, INTERDISCIPLINARY APPLICATIONS
COMPUTER SCIENCE, SOFTWARE ENGINEERING
COMPUTER SCIENCE, THEORY & METHODS
CONSTRUCTION & BUILDING TECHNOLOGY
CRIMINOLOGY & PENOLOGY
CRYSTALLOGRAPHY
DENTISTRY, ORAL SURGERY & MEDICINE
ECOLOGY
ECONOMICS
EDUCATION & EDUCATIONAL RESEARCH
EDUCATION, SCIENTIFIC DISCIPLINES
ELECTROCHEMISTRY
ENERGY & FUELS
ENGINEERING, AEROSPACE
ENGINEERING, BIOMEDICAL
ENGINEERING, CHEMICAL
ENGINEERING, CIVIL
ENGINEERING, ELECTRICAL & ELECTRONIC
ENGINEERING, ENVIRONMENTAL
ENGINEERING, GEOLOGICAL
ENGINEERING, INDUSTRIAL
ENGINEERING, MANUFACTURING
ENGINEERING, MECHANICAL
ENGINEERING, MULTIDISCIPLINARY
ENGINEERING, OCEAN
ENGINEERING, PETROLEUM
ENVIRONMENTAL SCIENCES
ENVIRONMENTAL STUDIES
EVOLUTIONARY BIOLOGY
FOOD SCIENCE & TECHNOLOGY
GASTROENTEROLOGY & HEPATOLOGY
GENETICS & HEREDITY
GEOCHEMISTRY & GEOPHYSICS
GEOLOGY
GEOSCIENCES, MULTIDISCIPLINARY
HEMATOLOGY
HISTORY & PHILOSOPHY OF SCIENCE
HOSPITALITY, LEISURE, SPORT & TOURISM
IMAGING SCIENCE & PHOTOGRAPHIC TECHNOLOGY
INSTRUMENTS & INSTRUMENTATION
INTERNATIONAL RELATIONS
LIMNOLOGY
MANAGEMENT
MATERIALS SCIENCE, BIOMATERIALS
MATERIALS SCIENCE, CERAMICS
MATERIALS SCIENCE, CHARACTERIZATION & TESTING
MATERIALS SCIENCE, COATINGS & FILMS
MATERIALS SCIENCE, COMPOSITES
MATERIALS SCIENCE, MULTIDISCIPLINARY
MATERIALS SCIENCE, TEXTILES
MATHEMATICAL & COMPUTATIONAL BIOLOGY
MATHEMATICS
MATHEMATICS, APPLIED
MATHEMATICS, INTERDISCIPLINARY APPLICATIONS
MECHANICS
MEDICAL INFORMATICS
MEDICINE, RESEARCH & EXPERIMENTAL
METALLURGY & METALLURGICAL ENGINEERING
METEOROLOGY & ATMOSPHERIC SCIENCES
MINERALOGY
MINING & MINERAL PROCESSING
MULTIDISCIPLINARY SCIENCES
MYCOLOGY
NANOSCIENCE & NANOTECHNOLOGY
NEUROSCIENCES
NUCLEAR SCIENCE & TECHNOLOGY
OCEANOGRAPHY
ONCOLOGY
OPERATIONS RESEARCH & MANAGEMENT SCIENCE
OPHTHALMOLOGY
OPTICS
ORNITHOLOGY
PERIPHERAL VASCULAR DISEASE
PHARMACOLOGY & PHARMACY
PHYSICS, APPLIED
PHYSICS, ATOMIC, MOLECULAR & CHEMICAL
PHYSICS, CONDENSED MATTER
PHYSICS, FLUIDS & PLASMAS
PHYSICS, MATHEMATICAL
PHYSICS, MULTIDISCIPLINARY
PHYSICS, NUCLEAR
PHYSICS, PARTICLES & FIELDS
PHYSIOLOGY
PLANNING & DEVELOPMENT
PLANT SCIENCES
POLITICAL SCIENCE
POLYMER SCIENCE
PSYCHOLOGY, EXPERIMENTAL
PSYCHOLOGY, MATHEMATICAL
PUBLIC, ENVIRONMENTAL & OCCUPATIONAL HEALTH
RADIOLOGY, NUCLEAR MEDICINE & MEDICAL IMAGING
REMOTE SENSING
RESPIRATORY SYSTEM
ROBOTICS
SOCIAL SCIENCES, INTERDISCIPLINARY
SOCIAL SCIENCES, MATHEMATICAL METHODS
SOCIOLOGY
SOIL SCIENCE
SPECTROSCOPY
SPORT SCIENCES
STATISTICS & PROBABILITY
SURGERY
TELECOMMUNICATIONS
THERMODYNAMICS
TRANSPORTATION SCIENCE & TECHNOLOGY
URBAN STUDIES
WATER RESOURCES

Most used definitions of fractional differentiation

Riemann–Liouville, 1920s:
(Letnikov, 1870s)

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}}, \qquad (n-1 \le \alpha < n)$$

Caputo, 1967:

For $f \in C^{(n)}[a,b], \quad f^{(k)}(a) = 0 \ (k = 0, \ldots, n-1)$
R-L, C, M-R and G-L definitions
are equivalent

Miller–Ross, 1990s
(Dzhrbashyan, 1960s)

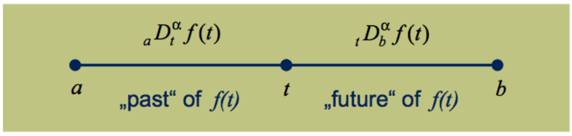
Grünwald–Letnikov, 1860s
(Liouville, 1830s)

$$D^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{n} f(t-kh)$$

Left- and right-sided derivatives

"Left - sided" : $${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}}$$

"Right - sided" : $${}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b \frac{f(\tau)\,d\tau}{(\tau-t)^{\alpha-n+1}}$$



Geometric interpretation of fractional integration: shadows on the walls

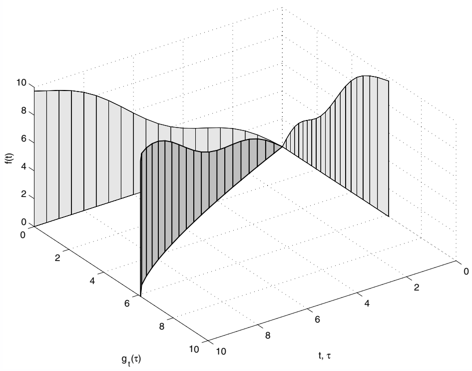
$${}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1}\,d\tau, \quad t \ge 0,$$

$${}_0I_t^\alpha f(t) = \int_0^t f(\tau)\,dg_t(\tau),$$

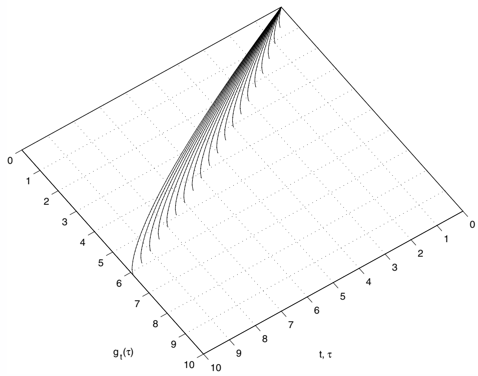
$$g_t(\tau) = \frac{1}{\Gamma(\alpha+1)}\{t^\alpha - (t-\tau)^\alpha\}.$$

For $t_1 = kt$, $\tau_1 = k\tau$ $(k>0)$ we have:

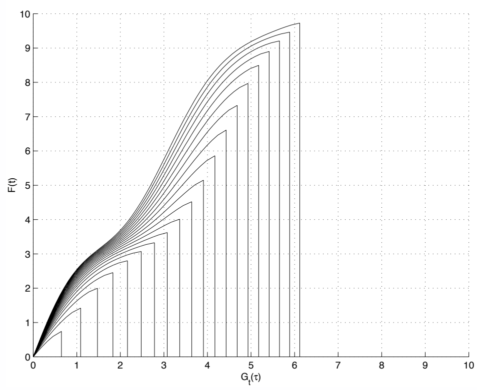
$$g_{t_1}(\tau_1) = g_{kt}(k\tau) = k^\alpha g_t(\tau).$$



“Live fence” and its shadows: ${}_0I_t^1 f(t)$ a ${}_0I_t^\alpha f(t)$, for $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, $0 \le t \le 10$.



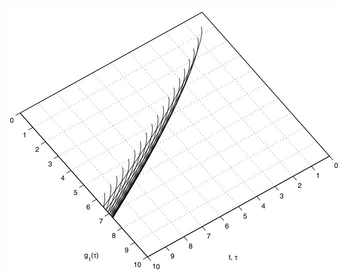
“Live fence”: basis shape is changing for ${}_0I_t^\alpha f(t)$, $\alpha = 0.75$, $0 \le t \le 10$.



Snapshots of the changing “shadow” of changing “fence” for ${}_0I_t^\alpha f(t)$, $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, with the time interval $\Delta t = 0.5$ between the snapshops.

Right-sided R-L integral

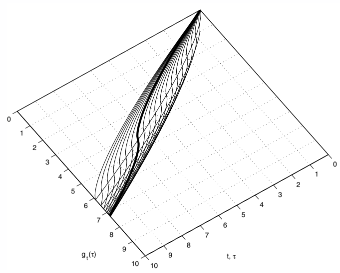
$${}_tI_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(\tau)(\tau-t)^{\alpha-1}\,d\tau, \quad t \le b,$$



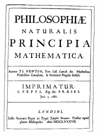
${}_tI_{10}^\alpha f(t)$, $\alpha = 0.75$, $0 \le t \le 10$

Riesz potential

$$_0R_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^b f(\tau)|\tau - t|^{\alpha-1}\, d\tau, \quad 0 \le t \le b,$$



$_0R_{10}^\alpha f(t)$, $\alpha = 0.75$, $0 \le t \le 10$



What is "time"?

I. Newton (*Principia Mat.*, 1686): introduced "mathematical time":

> "Absolute, true and mathematical time of itself, and from its own nature, flows equably without relation to anything external."

G. J. Whitrow (*The Natural Philosophy of Time*, 1961):

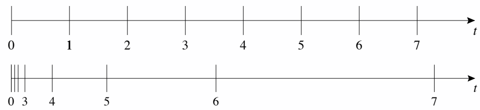
> "The outstanding mathematical achievement associated with the geometrization of time was, of course, the invention of the calculus of fluxions by Newton."
> "Mathematically, Newton seems to have found support for his belief in absolute time by the need, in principle, for an ideal rate-measurer."

How do we measure time intervals?

Only by observing some processes, which we consider as regularly repeated. G. Clemence (*Amer. Scientist*, vol. 40, 1952) wrote:

> "The measurement of time is essentially a process of counting. Any recurring phenomenon whatever, the occurences of which can be counted, is in fact a measure of time."

What is "time"?



Computing the distance when the time is slowing down

Person N individual "seconds"	Recorded values of velocity [m/s]	Observer O absolute (cosmic) "seconds"
0	10	0
1	11	1
2	12	3
3	13	7
4	12	15
5	11	31
6	10	63
7	9	127

$D_N = 10\cdot 1 + 11\cdot 1 + 12\cdot 1 + 13\cdot 1 + 12\cdot 1 + 11\cdot 1 + 10\cdot 1 = 79.$

$D_O = 10\cdot 1 + 11\cdot 2 + 12\cdot 4 + 13\cdot 8 + 12\cdot 16 + 11\cdot 32 + 10\cdot 64 = 1368.$

Physical interpretation of Stieltjes integral (1)

Imagine a car equipped with two devices for measurements: the speedometer recording the velocity $v(\tau)$, and the clock which should show the time τ. The clock, however, shows the time incorrectly.

Suppose that the relationship between the wrong time τ, shown by the clock and considered by the driver A as the correct time, and the true time T, on the other, is described by the function $T = g(\tau)$.

Physical interpretation of Stieltjes integral (2)

Driver A's computations:

$$S_A(t) = \int_0^t v(\tau)d\tau\,.$$

Well-informed observer O's computations:

$$S_O(t) = \int_0^t v(\tau)dg(\tau).$$

This example shows that the Stieltjes integral can be interpreted as the real distance passed by a moving object, for which we have recorded correct values of speed and incorrect values of time; the relationship between the wrongly recorded time τ and the correct time T is given by a known function $T = g(\tau)$.

Physical interpretation of the Riemann-Liouville integral: "shadows of the past"

$$S_O(t) = \int_0^t v(\tau)\, dg_t(\tau) = {}_0I_t^\alpha v(t),$$

$$g_t(\tau) = \frac{1}{\Gamma(\alpha+1)}\{t^\alpha - (t-\tau)^\alpha\}.$$

The left-sided Riemann–Liouville fractional integral of the individual speed $v(\tau)$ of a moving object, for which the relationship between its individual time τ and the cosmic time T at each individual time instance t is given by the known function $T = g_t(\tau)$, represents the real distance $S_O(t)$ passed by that object.

Interpretations of the Volterra integrals

$$K * f(t) = \int_0^t f(\tau)k(t-\tau)d\tau$$

Assuming that $k(t) = K'(t)$, this integral takes the form:

$$K * f(t) = \int_0^t f(\tau)dq_t(\tau),$$

$$q_t(\tau) = K(t) - K(t-\tau).$$

The geometric and physical interpretations of the Volterra convolution integral are then similar to the suggested interpretations for fractional integrals.

Physical interpretation of initial conditions for fractional differential equations with the Riemann-Liouville fractional derivatives

Recall the Laplace transform method for fractional differential equations

$$\int_0^\infty e^{-st}\, {}_0D_t^\alpha f(t)\, dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k \left[{}_0D_t^{\alpha-k-1} f(t)\right]_{t=0},$$

$$(n-1 < \alpha \le n).$$

$$t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm z t^\alpha) \leftrightarrows \frac{k!\, p^{\alpha-\beta}}{(p^\alpha \mp a)^{k+1}}$$

The Laplace transform method for FDEs

EXAMPLE 1.

$${}_0D_t^{1/2} f(t) + a f(t) = 0, \quad (t > 0); \qquad \left[{}_0D_t^{-1/2} f(t)\right]_{t=0} = C$$

Solution: applying the Laplace transform we obtain

$$F(s) = \frac{C}{s^{1/2}+a}, \qquad C = \left[{}_0D_t^{-1/2} f(t)\right]_{t=0}$$

and the inverse Laplace transform gives the solution:

$$f(t) = Ct^{-1/2} E_{\frac{1}{2},\frac{1}{2}}(-a\sqrt{t}).$$

If a=1, then

$$f(t) = C\left(\frac{1}{\sqrt{\pi t}} - e^t \operatorname{erfc}(\sqrt{t})\right)$$

The Laplace transform method for FDEs

EXAMPLE 2.

$${}_0D_t^\alpha y(t) - \lambda y(t) = h(t), \quad (t > 0); \qquad n-1 < \alpha < n.$$

$$\left[{}_0D_t^{\alpha-k} y(t)\right]_{t=0} = b_k, \quad (k = 1, 2, \ldots, n)$$

Solution: applying the Laplace transform:

$$s^\alpha Y(s) - \lambda Y(s) = H(s) + \sum_{k=1}^n b_k s^{k-1},$$

$$Y(s) = \frac{H(s)}{s^\alpha - \lambda} + \sum_{k=1}^n b_k \frac{s^{k-1}}{s^\alpha - \lambda}$$

and the inverse transform gives the solution:

$$y(t) = \sum_{k=1}^n b_k t^{\alpha-k} E_{\alpha,\alpha-k+1}(\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) h(\tau) d\tau$$

We deal with the Riemann-Liouville derivatives ($n-1 \le \alpha < n$):

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(\tau)\, d\tau}{(t-\tau)^{\alpha-n+1}}. $$

Fractional differential equations in terms of RL derivatives require initial conditions expressed in terms of initial values of fractional derivatives of the unknown function.

A typical initial value problem ($n-1 < \alpha < n$):

$$ {}_0D_t^{\alpha} f(t) + a f(t) = h(t); \qquad (t > 0) $$

$$ \left[{}_0D_t^{\alpha-k} f(t) \right]_{t \to 0} = b_k, \qquad (k = 1, 2, \ldots, n). $$

What about initial conditions?

$$ \left[{}_0D_t^{\alpha-k} y(t) \right]_{t=0} = b_k, \qquad (k = 1, \ldots, [\alpha] + 1) $$

K. Diethelm, N. J. Ford, A. D. Freed, and Yu. Luchko (January 2005):

> "A typical feature of differential equations (both classical and fractional) is the need to specify additional conditions in order to produce a unique solution. For the case of Caputo FDEs, these additional conditions are just the static initial conditions ..., which are akin to those of classical ODEs, and are therefore familiar to us. In contrast, for Riemann-Liouville FDEs, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point $x = 0$ ..., which are functions of x. These initial conditions are not physical; furthermore, it is not clear how such quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis."

N. Heymans and I. Podlubny:

> "Physical interpretation of initial conditions for fractional differential equations with the Riemann-Liouville fractional derivatives", *Rheologica Acta*, vol. 45, no. 5, June 2006, pp. 765–772.

Spring-pot model

Spring-pot is a linear viscoelastic element whose behaviour is intermediate between that of elastic element (spring) and a viscous element (dashpot). The term "spring-pot" was introduced by Koeller (1984), although the concept of an element with intermediate properties had been introduced some time earlier (G. W. Scott Blair, 1930s-40s). The constitutive equation of a spring-pot is:

$$ \sigma(t) = K\, {}_0D_t^{\alpha} \epsilon(t) \qquad \text{or} \qquad \epsilon(t) = \frac{1}{K}\, {}_0D_t^{-\alpha} \sigma(t) $$

Spring-pot model: Creep

A stress step σ_0 is applied at initial time $t = 0$. The change of $\epsilon(t)$ is described by the FDE

$$ {}_0D_t^{\alpha} \epsilon(t) = \frac{\sigma_0}{K} $$

An initial condition involving ${}_0D_t^{\alpha-1}\epsilon(t)$ is required. It can be found by taking the first-order integral of the constitutive equation and letting $t \to 0$

$$ \left[{}_0D_t^{\alpha-1} \epsilon(t) \right]_{t \to 0} = \left[{}_0D_t^{-1} (\sigma_0 / K) \right]_{t \to 0}. $$

In the considered case stress is finite at all times, therefore the required IC is

$$ \left[{}_0D_t^{\alpha-1} \epsilon(t) \right]_{t \to 0} = 0. $$

Spring-pot model: Impulse response

An impulse of stress defined as $B\delta(t)$ applied to the spring-pot at time $t = 0$. After that, the stress remains zero. The strain $\epsilon(t)$ for $t > 0$ is the solution of FDE

$$ {}_0D_t^{\alpha} \epsilon(t) = 0. $$

An initial condition involving $\left[{}_0D_t^{\alpha-1} \epsilon(t) \right]_{t \to 0}$ is required.

This can be found through integration of the constitutive equation, as

$$ \left[{}_0D_t^{\alpha-1} \epsilon(t) \right]_{t \to 0} = \left[{}_0D_t^{-1} \sigma(t) / K \right]_{t \to 0}, $$

which gives the following initial condition:

$$ \left[{}_0D_t^{\alpha-1} \epsilon(t) \right]_{t \to 0} = B / K. $$

The key: look for inseparable twins

In a general case, when we consider some FDE for, say, $U(t)$, we have to consider also some function $V(t)$, for which some *dual relation* exists between $U(t)$ and $V(t)$. For example: stress $\sigma(t)$ and strain $\epsilon(t)$ in viscoelasticity; charge $q(t)$ and voltage $v(t)$ in electrical circuits; temperature difference $T(t)$ and the heat flux $q(t)$ in heat conduction; etc. Functions $U(t)$ and $V(t)$ are normally related by some basic physical law for the particular field of science.

In each scientific field there are such pairs of functions like the aforementioned, which are as *inseparable as Siamese twins*: the left-hand side of the initial condition involves one of them, whereas the evaluation of the right-hand side is related to the other.