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#2
Numerical fractional differentiation

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How do we do steps forward?

G. W. Leibniz (1695–1697)

In the letters to J. Wallis and J. Bernulli (in 1697) Leibniz mentioned the possible approach to fractional-order differentiation in that sense, that for non-integer values of n the definition could be the following:

$$\frac{d^n e^{mx}}{dx^n} = m^n e^{mx},$$

L. Euler (1730)

$$\frac{d^n x^m}{dx^n} = m(m-1)\ \ldots\ (m-n+1)x^{m-n}$$

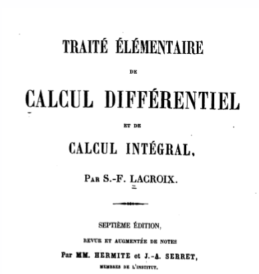
$$\Gamma(m+1) = m(m-1)\ \ldots\ (m-n+1)\,\Gamma(m-n+1)$$

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}.$$

Euler suggested to use this relationship also for negative or non-integer (rational) values of n. Taking $m = 1$ and $n = \frac{1}{2}$, Euler obtained:

$$\frac{d^{1/2} x}{dx^{1/2}} = \sqrt{\frac{4x}{\pi}} \qquad \left(= \frac{2}{\sqrt{\pi}} x^{1/2} \right)$$

S. F. Lacroix adopted Euler's derivation for his successful textbook (*Traité du Calcul Différentiel et du Calcul Intégral*, Courcier, Paris, t. 3, 1819; pp. 409–410).


TRAITÉ ÉLÉMENTAIRE
DE
CALCUL DIFFÉRENTIEL
ET DE
CALCUL INTÉGRAL,
PAR S.-F. LACROIX.
SEPTIÈME ÉDITION,
REVUE ET AUGMENTÉE DE NOTES
Par MM. HERMITE et J.-A. SERRET,
MEMBRES DE L'INSTITUT.

J. B. J. Fourier (1820–1822)

The first step to generalization of the notion of differentiation for **arbitrary functions** was done by J. B. J. Fourier (*Théorie Analytique de la Chaleur*, Didot, Paris, 1822; pp. 499–508).

After introducing his famous formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(px - pz)dp,$$

Fourier made a remark that

$$\frac{d^n f(x)}{dx^n} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(px - pz + n\frac{\pi}{2})dp,$$

and this relationship could serve as a definition of the n-th order derivative for non-integer n.

N. H. Abel (1823–1826)

N. H. Abel: Solution de quelques problèmes à l'aide d'intégrales définies (1823). *Œuvres complètes de Niels Henrik Abel*, vol. 1, Grondahl, Christiania, 1881, pp. 11–18.

In fact, Abel solved the equation

Simply a Caputo derivative

$$\int_0^x \frac{s'(\eta)d\eta}{(x-\eta)^\alpha} = \psi(x), \qquad \Gamma(1-\alpha)\frac{d^\alpha}{dx^\alpha}s(x) = \psi(x)$$

for an arbitrary α (and not just for $\alpha = \frac{1}{2}$):

$$s(x) = \frac{\sin(\pi\alpha)}{\pi}x^\alpha \int_0^1 \frac{\psi(xt)dt}{(1-t)^{1-\alpha}}.$$

After that, Abel *expressed* the obtained solution with the help of an integral of order α:

$$s(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d^{-\alpha}\psi(x)}{dx^{-\alpha}}$$

J. Liouville (1832–1855)

Three approaches:

I. Following Leibniz:

$$\frac{d^m e^{ax}}{dx^n} = a^m e^{ax},$$

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x},$$

$$\frac{d^\nu f(x)}{dx^\nu} = \sum_{n=0}^{\infty} c_n a_n^\nu e^{a_n x}$$

J. Liouville (1832–1855)

Three approaches:

II. *Integrals* of non-integer order:

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{(-1)^\mu\Gamma(\mu)}\int_0^\infty \Phi(x+\alpha)\alpha^{\mu-1}d\alpha$$

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{\Gamma(\mu)}\int_0^\infty \Phi(x-\alpha)\alpha^{\mu-1}d\alpha$$

or (after the substitution $\tau = x+\alpha$, $\tau = x-\alpha$)

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{(-1)^\mu\Gamma(\mu)}\int_x^\infty (\tau-x)^{\mu-1}\Phi(\tau)d\tau$$

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{\Gamma(\mu)}\int_{-\infty}^x (x-\tau)^{\mu-1}\Phi(\tau)d\tau.$$

J. Liouville (1832–1855)

Three approaches:

III. *Derivatives* of non-integer order:

$$\frac{d^\mu F(x)}{dx^\mu} = \frac{(-1)^\mu}{h^\mu}\Big(F(x) - \frac{\mu}{1}F(x+h) + \frac{\mu(\mu-1)}{1\cdot 2}F(x+2h) - \dots\Big)$$

$$\frac{d^\mu F(x)}{dx^\mu} = \frac{1}{h^\mu}\Big(F(x) - \frac{\mu}{1}F(x-h) + \frac{\mu(\mu-1)}{1\cdot 2}F(x-2h) - \dots\Big).$$

(Equality is in the sense $\lim_{h\to 0}$).

Liouville was the first, who realized the possibility of considereation of *left-sided* and *right-sided* fractional integrals and derivatives.

G. F. B. Riemann (1847; 1876)

Riemann used a generalization of the Taylor series for obtaining a formula for fractional-order integration, which is given below in contemporary notation:

$$D^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_c^x (x-t)^{\nu-1}f(t)dt + \psi(t)$$

Riemann introduced an arbitrary (complementary) function $\psi(x)$ because he did not fixed the lower bound of integration c – a disadvantage, which cannot be removed in the framework of his approach.

N. Ya. Sonin (1869)
A. V. Letnikov (1872)
H. Laurent (1884)
N. Nekrasov (1888)
K. Nishimoto (1987–)

Cauchy's formula:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(t)}{(t-z)^{n+1}}dt$$

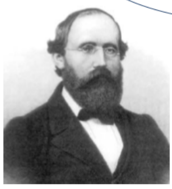
For non-integer $n = \nu$ a branch point of the function $(t-z)^{-\nu-1}$ appears instead of a pole:

$$D^\nu f(z) = \frac{\Gamma(\nu+1)}{2\pi i}\int_c^{x^+} \frac{f(t)}{(t-z)^{\nu+1}}dt$$

Riemann–Liouville definition

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}} $$

$$ (n-1 \le \alpha < n) $$



G.F.B. Riemann (1826–1866)

J. Liouville (1809–1882)

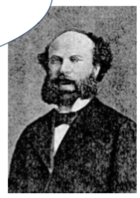
Grünwald–Letnikov definition

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh) $$

$[x]$ – integer part of x

A.K. Grünwald

A.V. Letnikov



A direct approach to numerical fractional differentiation

Some notation

$$ f_N \equiv f(0), $$

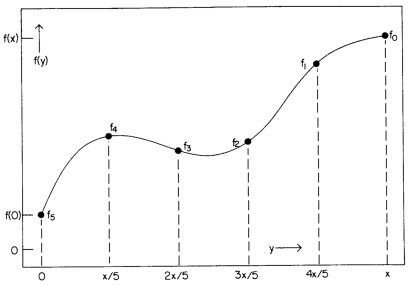
$$ f_{N-1} \equiv f\left(\frac{x}{N}\right), $$

$$ \vdots $$

$$ f_j \equiv f\left(x - \frac{jx}{N}\right), $$

$$ \vdots $$

$$ f_0 \equiv f(x) $$



Notice: backwards numbering is convenient.

The G1 algorithm

Recall the Grunwald-Letnikov definition:

$[x]$ means the integer part of x

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{{}_a\Delta_h^\alpha f(t)}{h^\alpha}, \qquad {}_a\Delta_h^\alpha f(t) = \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh). $$

Omitting the limit gives the simplest approximation:

$$ {}_aD_t^\alpha f(t) \approx {}_a\Delta_h^\alpha f(t) $$

The G1 algorithm

Omitting the limit in the Grunwald-Letnikov definition gives the simplest formula for approximate calculations:

$$ {}_aD_t^\alpha f(x) \approx \left\{ \frac{\left[\frac{x-a}{N}\right]^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\left[\frac{x-a}{N}\right]\right) \right\} $$

For $a = 0$ we have:

$$ {}_0D_t^\alpha f(x) \approx \frac{x^{-q}N^q}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f_j $$

Computation of binomial coefficients
Using recurrence relationships

For the implementation we need to compute

$$w_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}, \qquad k = 0, 1, 2, \ldots$$

The recurrence relationships can be used:

$$w_0^{(\alpha)} = 1; \quad w_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{k}\right) w_{k-1}^{(\alpha)}, \quad k = 1, 2, 3, \ldots$$

Computation of binomial coefficients
Using FFT (fast Fourier transform)

The binomial coefficients can be considered as the coefficients of the power series expansion

$$(1-z)^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k = \sum_{k=0}^{\infty} w_k^{(\alpha)} z^k$$

Taking $z = e^{-i\varphi}$ we obtain

$$(1 - e^{-i\varphi})^\alpha = \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{-ik\varphi}$$

Computation of binomial coefficients
Using FFT (fast Fourier transform)

We have (from the previous slide):

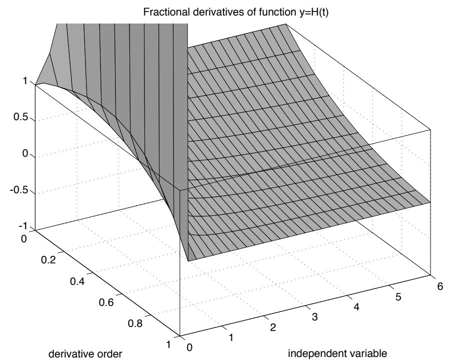
$$(1 - e^{-i\varphi})^\alpha = \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{-ik\varphi}$$

Therefore, the coefficients $w_k^{(\alpha)}$ can be considered as Fourier transforms:

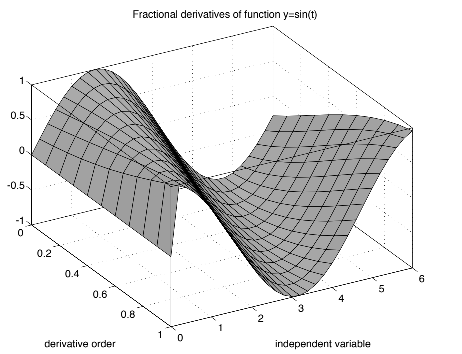
$$w_k^{(\alpha)} = \frac{1}{2\pi} \int_0^{2\pi} f_\alpha(\varphi) e^{ik\varphi} d\varphi, \qquad f_\alpha(\varphi) = (1 - e^{-i\varphi})^\alpha.$$

The fast Fourier transform (FFT) can be used here!

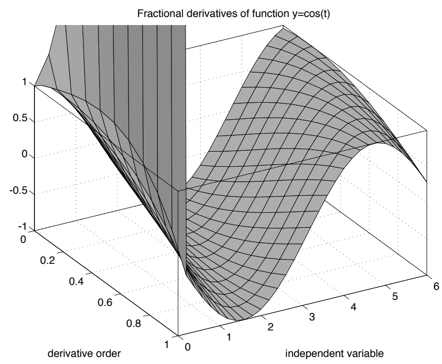
Using G1 for computations



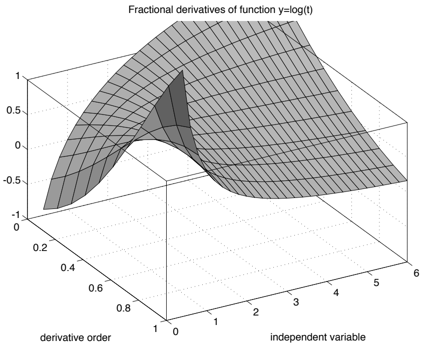
Using G1 for computations



Using G1 for computations



Using G1 for computations



Notice: infinities at t=0 are removed to make the rest of the picture visible.

Higher-order approximations

We have seen the first-order approximation:

$$ {}_0D_t^{\alpha} f(x) \approx h^{-\alpha} \sum_{k=0}^{[t/h]} w_k^{(\alpha)} f(t-kh) $$

The weights $w_k^{(\alpha)}$, $(k = 0, 1, 2, \ldots, n, \; n = [\frac{t}{h}])$ assigned to the values $f(t\text{-}jh)$ are the first n+1 coefficients of the Taylor series expansion of the function

$$ \omega_1^{(\alpha)}(z) = (1-z)^{\alpha} = \left(\omega_1(z)\right)^{\alpha} $$

Higher-order approximations

Christian Lubich's formulas:

$$ \omega_2^{(\alpha)}(z) = (\frac{3}{2} - 2z + \frac{1}{2}z^2)^{\alpha}, $$

$$ \omega_3^{(\alpha)}(z) = (\frac{11}{6} - 3z + \frac{3}{2}z^2 - \frac{1}{3}z^2)^{\alpha}, $$

$$ \omega_4^{(\alpha)}(z) = (\frac{25}{12} - 4z + 3z^2 - \frac{4}{3}z^3 + \frac{1}{4}z^4)^{\alpha}, $$

$$ \omega_5^{(\alpha)}(z) = (\frac{137}{60} - 5z + 5z^2 - \frac{10}{3}z^3 + \frac{5}{4}z^4 - \frac{1}{5}z^5)^{\alpha}, $$

$$ \omega_6^{(\alpha)}(z) = (\frac{147}{60} - 6z + \frac{15}{2}z^2 - \frac{20}{3}z^3 + \frac{15}{4}z^4 - \frac{6}{5}z^5 + \frac{1}{6})^{\alpha} $$

Expand these functions in Taylor series and use the coefficients as weights in the formula

$$ {}_0D_t^{\alpha} f(t) \approx h^{-\alpha} \sum_{k=0}^{[t/h]} w_k \; f(t-kh) $$

The G2 algorithm

Oldham and Spanier (1974) observed that the approximations

$$ {}_aD_t^{-1} f(x) = \lim_{N\to\infty} \left\{ \frac{x-a}{N} \sum_{j=0}^{N-1} f\left(x - [j+\tfrac{1}{2}]\left[\frac{x-a}{N}\right]\right) \right\} $$

$$ {}_aD_t^{1} f(x) = \lim_{N\to\infty} \left\{ \left[\frac{x-a}{N}\right]^{-1} \sum_{j=0}^{1} [-]^j f\left(x - [j-\tfrac{1}{2}]\left[\frac{x-a}{N}\right]\right) \right\} $$

give faster convergence, and suggested "fractional central differences"

$$ {}_aD_t^{q} f(x) = \lim_{N\to\infty} \left\{ \frac{\left[\frac{x-a}{N}\right]^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - [j-\tfrac{1}{2}q]\left[\frac{x-a}{N}\right]\right) \right\} $$

The G2 algorithm

Taking lower terminal $a = 0$ we have

$$ {}_0D_t^{q} f(x) = \lim_{N\to\infty} \left\{ \frac{x^{-q}N^{q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x + \frac{qx}{2N} - \frac{jx}{N}\right) \right\} $$

This formula uses the function values other than at nodes,
so we have to interpolate (Langrange three point interpolation):

$$ f\left(x + \frac{qx}{2N} - \frac{jx}{N}\right) \approx \left[\frac{q}{4} + \frac{q^2}{8}\right] f\left(x + \frac{x}{N} - \frac{jx}{N}\right) + \left[1 - \frac{q^2}{4}\right] f\left(x - \frac{jx}{N}\right) + \left[\frac{q^2}{8} - \frac{q}{4}\right] f\left(x - \frac{x}{N} - \frac{jx}{N}\right) $$

This gives the G2 algorithm:

$$ {}_0D_t^{q} f(x) = \frac{x^{-q}N^{q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} \{f_j + \tfrac{1}{4}q[f_{j-1} - f_{j+1}] + \tfrac{1}{8}q^2[f_{j-1} - 2f_j + f_{j+1}]\} $$

The R1 algorithm

G1 and G2 are based on fractional differences.
R1 and R2 are based on approximation of integration.

Take $q < 0$ (fractional integration):

$$ {}_0D_t^{q} f(x) = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(y)\,dy}{[x-y]^{q+1}} = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(x-y)\,dy}{y^{q+1}} $$

useful form

$$ = \frac{1}{\Gamma(-q)} \sum_{j=0}^{N-1} \int_{jx/N}^{[jx+x]/N} \frac{f(x-y)\,dy}{y^{q+1}} $$

The R1 algorithm

Using the approximation

$$\int_{jx/N}^{[jx+x]/N} \frac{f(x-y)\,dy}{y^{q+1}} \approx \frac{f\left(x-\frac{jx}{N}\right)+f\left(x-\frac{x}{N}-\frac{jx}{N}\right)}{2} \int_{jx/N}^{[jx+x]/N} \frac{dy}{y^{q+1}}$$

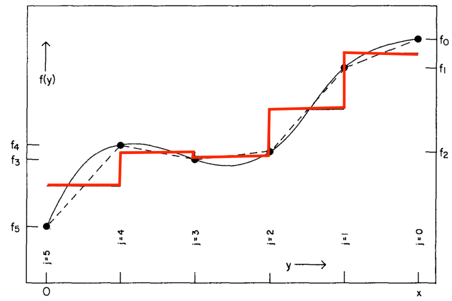
$$= \frac{f_j+f_{j+1}}{-2q}\left\{\left[\frac{jx+x}{N}\right]^{-q} - \left[\frac{jx}{N}\right]^{-q}\right\},$$

we obtain the R1 algorithm:

$${}_0D_t^q f(x) = \frac{x^{-q}N^q}{\Gamma(1-q)} \sum_{j=0}^{N-1} \frac{f_j+f_{j+1}}{2}\{[j+1]^{-q}-j^{-q}\} \qquad (\text{ for } q<0)$$

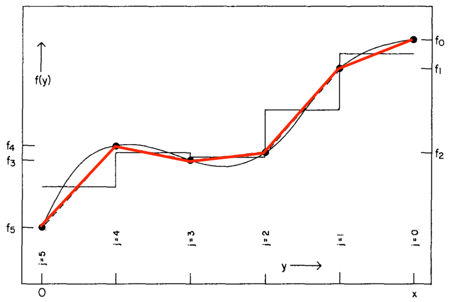
The R1 algorithm

In the previous slide we have used the piecewise constant approximation of the function using the function values in the middle of the subintervals:



The R2 algorithm

A better function approximation can be achieved using piecewise linear continuos function:



The R2 algorithm

Considering the piecewise linear approximation we have:

$$\int_{jx/N}^{[jx+x]/N} \frac{f(x-y)\,dy}{y^{q+1}} \approx \int_{jx/N}^{[jx+x]/N} \frac{\left\{\left[1+j-\frac{Ny}{x}\right] f\left(x-\frac{jx}{N}\right)+\left[\frac{Ny}{x}-j\right] f\left(x-\frac{x}{N}-\frac{jx}{N}\right)\right\} dy}{y^{q+1}}$$

and this gives the R2 algorithm:

$${}_0D_t^q f(x) = \frac{x^{-q}N^q}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{[j+1]f_j - jf_{j+1}}{-q}\{[j+1]^{-q}-j^{-q}\} + \frac{f_{j+1}-f_j}{1-q}\{[j+1]^{1-q}-j^{1-q}\}$$

(for $q<0$)

Notice: both R1 and R2 algorithms allow consideration of non-equidistant discretization nodes.

The L1 algorithm

Both R1 and R2 are restricted to negatives orders (integration only). How to extend beyond this "wall"?

Let us take $0 \leqq q < 1$. Then

Caputo

$${}_0D_t^q f(x) = \frac{x^{-q}f(0)}{\Gamma(1-q)} + \frac{1}{\Gamma(1-q)} \int_0^x \left[\frac{df}{dy}(y)\right] \frac{dy}{[x-y]^q}$$

$$= \frac{1}{\Gamma(1-q)}\left\{\frac{f(0)}{x^q} + \sum_{j=0}^{N-1} \int_{jx/N}^{[jx+x]/N} \left[\frac{df}{dy}(x-y)\right] \frac{dy}{y^q}\right\}$$

The L1 algorithm

Now let us approximate each term using

$$\int_{jx/N}^{[jx+x]/N} \left[\frac{df}{dy}(x-y)\right] \frac{dy}{y^q} \approx \frac{f\left(x-\frac{jx}{N}\right)-f\left(x-\frac{x}{N}-\frac{jx}{N}\right)}{x/N} \int_{jx/N}^{[jx+x]/N} \frac{dy}{y^q}$$

$$= \frac{x^{-q}N^q}{1-q}[f_j-f_{j+1}][(j+1)^{1-q}-j^{1-q}]$$

This leads to the LI algorithm:

$${}_0D_t^q f(x) = \frac{x^{-q}N^q}{\Gamma(2-q)}\left[\frac{[1-q]f_N}{N^q} + \sum_{j=0}^{N-1}[f_j-f_{j+1}][(j+1)^{1-q}-j^{1-q}]\right]$$

$0 \leq q < 1$

The L2 algorithm

Similarly, taking $1 \leqq q < 2$ we can write

$$ {}_0D_t^q f(x) = \frac{x^{-q} f(0)}{\Gamma(1-q)} + \frac{x^{1-q} f^{(1)}(0)}{\Gamma(2-q)} + \frac{1}{\Gamma(2-q)} \int_0^x \frac{f^{(2)}(y)\,dy}{[x-y]^{q-1}} $$

$$ = \frac{1}{\Gamma(2-q)} \left[\frac{[1-q] f(0)}{x^q} + \frac{f^{(1)}(0)}{x^{q-1}} + \sum_{j=0}^{N-1} \int_{jx/N}^{[jx+x]/N} \frac{f^{(2)}(x-y)\,dy}{y^{q-1}} \right] $$

We need some approximations for the first and second order derivatives here.

The L2 algorithm

Using the approximations

$$ f^{(1)}(0) \approx \frac{f\left(\frac{x}{N}\right) - f(0)}{x/N} = \frac{N}{x}[f_{N-1} - f_N] $$

$$ \int_{jx/N}^{[jx+x]/N} f^{(2)}(x-y) \frac{dy}{y^{q-1}} \approx \frac{f\left(x + \frac{x}{N} - \frac{jx}{N}\right) - 2f\left(x - \frac{jx}{N}\right) + f\left(x - \frac{x}{N} - \frac{jx}{N}\right)}{[x/N]^2} \int_{jx/N}^{[jx+x]/N} \frac{dy}{y^{q-1}} $$

$$ = \frac{x^{-q} N^q}{2-q} [f_{j-1} - 2f_j + f_{j+1}][(j+1)^{2-q} - j^{2-q}] $$

we obtain the L2 algorithm

$$ {}_0D_t^q f(x) = \frac{x^{-q} N^q}{\Gamma(3-q)} \left[\frac{[1-q][2-q] f_N}{N^q} + \frac{[2-q][f_{N-1} - f_N]}{N^{q-1}} + \sum_{j=0}^{N-1} [f_{j-1} - 2f_j + f_{j+1}][(j+1)^{2-q} - j^{2-q}] \right] $$

$1 \leqq q < 2$

The D algorithm

Kai Diethelm suggested (1997) numerical evaluation of fractional derivatives using quadrature formulas for finite-part integrals. First, use change of variables to transform the interval $[0, T]$ to $[0,1]$ and introduce an equidistant grid with nodes $t_j = j/m$. Then

$$ {}_0D_{t_j}^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^{t_j} \frac{f(\tau)d\tau}{(t_j - \tau)^{\alpha+1}} = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \int_0^1 \frac{f(t_j - t_j \xi)\,d\xi}{\xi^{\alpha+1}} $$

this singularity requires the use of finite part integrals

and the D algorithm is:

$$ {}_0D_{t_j}^{\alpha} f(t) \approx \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \sum_{k=0}^{j} \omega_{kj} f\left(\frac{k}{j}\right), \quad \omega_{kj} = \frac{j^{\alpha}}{\alpha(1-\alpha)} \begin{cases} -1, & \text{for } k = 0, \\ 2k^{1-\alpha} - (k-1)^{1-\alpha} - (k+1)^{1-\alpha}, & \text{for } k = 1, 2, \ldots, j-1, \\ (\alpha-1)k^{-\alpha} - (k-1)^{1-\alpha} + k^{1-\alpha}, & \text{for } k = j. \end{cases} $$

$(j = 1, 2, \ldots, m)$

A general framework

All considered algorithms (G1, G2, R1, R2, L1, L2, D) can be written in the same form as

$$ {}_0D_t^q f(x) = \frac{N^q}{x^q} \sum_{j=-1}^{N} w_j(q) f_j $$

G1 in brief:

Typical $w_j(q)$	$\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}$
Range of typicality	$0 \leqq j \leqq N-1$
Values of j for which $w_j(q) = 0$	$-1, N$
Atypical values	none

G2 in brief:

Typical $w_j(q)$	$\frac{\Gamma(j-1-q)}{\Gamma(-q)\Gamma(j+2)} \left\{ j^2 - \frac{jq}{2}[q+3] - [q+1]\left[\frac{q^3}{8} + \frac{q^2}{2} - 1\right] \right\}$
Range of typicality	$0 \leqq j \leqq N-1$
Values of j for which $w_j(q) = 0$	none
Atypical values	$w_{-1}(q) = \frac{q^2 + 2q}{8\Gamma(-q)}$ $w_N(q) = \frac{\Gamma(N-1-q)}{\Gamma(-q)\Gamma(N)}\left[\frac{q^2}{8} - \frac{q}{4}\right]$

R1 in brief:

Typical $w_j(q)$	$\dfrac{[j+1]^{-q} - [j-1]^{-q}}{\Gamma(1-q)}$
Range of typicality	$1 \leqq j \leqq N-1$
Values of j for which $w_j(q) = 0$	-1
Atypical values	$w_0(q) = \dfrac{1}{2\Gamma(1-q)}$ $\quad w_N(q) = \dfrac{N^{-q} - [N-1]^{-q}}{2\Gamma(1-q)}$

R2 in brief:

Typical $w_j(q)$	$\dfrac{[j+1]^{1-q} - 2j^{1-q} + [j-1]^{1-q}}{\Gamma(2-q)}$
Range of typicality	$1 \leqq j \leqq N-1$
Values of j for which $w_j(q) = 0$	-1
Atypical values	$w_0(q) = 1/\Gamma(2-q)$ $w_N(q) = \dfrac{[N-1]^{1-q} - N^{1-q} + [1-q]N^{-q}}{\Gamma(2-q)}$

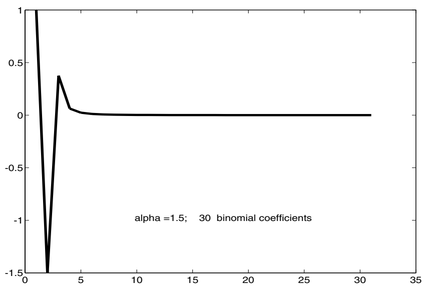
L1 in brief:

Typical $w_j(q)$	$\dfrac{[j+1]^{1-q} - 2j^{1-q} + [j-1]^{1-q}}{\Gamma(2-q)}$
Range of typicality	$1 \leqq j \leqq N-1$
Values of j for which $w_j(q) = 0$	-1
Atypical values	$w_0(q) = 1/\Gamma(2-q)$ $w_N(q) = \dfrac{[N-1]^{1-q} - N^{1-q} + [1-q]N^{-q}}{\Gamma(2-q)}$

L2 in brief:

Typical $w_j(q)$	$\dfrac{[j+2]^{2-q} - 3[j+1]^{2-q} + 3j^{2-q} - [j-1]^{2-q}}{\Gamma(3-q)}$
Range of typicality	$1 \leqq j \leqq N-2$
Values of j for which $w_j(q) = 0$	none
Atypical values	$w_{-1}(q) = 1/\Gamma(3-q)$ $w_0(q) = \dfrac{2^{2-q} - 3}{\Gamma(3-q)}$ $w_{N-1}(q) = [\Gamma(3-q)]^{-1}\{[2-q]N^{1-q} - 2N^{2-q} + 3[N-1]^{2-q} - [N-2]^{2-q}\}$ $w_N(q) = [\Gamma(3-q)]^{-1}\{[1-q][2-q]N^{-q} - [2-q]N^{1-q} + N^{2-q} - [N-1]^{2-q}\}$

The "short memory" principle



Behavior of coefficients in GI

$${}_aD_t^{\alpha} f(t) \approx {}_{t-L}D_t^{\alpha} f(t), \qquad (t > a + L)$$

"Memory length" depends on required accuracy

The "short memory" principle

$${}_aD_t^{\alpha} f(t) \approx {}_{t-L}D_t^{\alpha} f(t), \qquad (t > a + L)$$

If $f(t) \leq M$ for $a \leq t \leq b$, then it can be shown that

$$\Delta(t) = |\, {}_aD_t^{\alpha} f(t) - {}_{t-L}D_t^{\alpha} f(t)\,| \leq \frac{ML^{-\alpha}}{|\Gamma(1-\alpha)|}, \qquad (a + L \leq t \leq b)$$

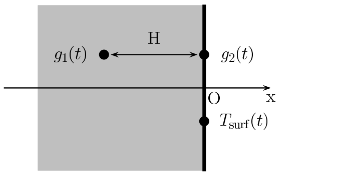
Therefore,

$$\Delta(t) \leq \epsilon, \quad (a + L \leq t \leq b), \qquad \text{if} \quad L \geq \left(\frac{M}{\epsilon\,|\Gamma(1-\alpha)|}\right)^{1/\alpha}$$

Sample applications

Fractional derivatives of order 1/2
can appear even within the context
of the integer-order models!

Heat flux in a blast furnace wall



$$\hat{c}\hat{\rho}\frac{\partial T}{\partial t} = \hat{\lambda}\frac{\partial^2 T}{\partial x^2}, \qquad (t > 0, \quad -\infty < x < 0)$$

$$T(0, x) = T_0$$

$$T(t, 0) = T_{\mathrm{surf}}(t)$$

$$\left| \lim_{x \to -\infty} T(t, x) \right| < \infty$$

Heat flux in a blast furnace wall

Introduce an auxiliary function

$$u(t, x) = T(t, x) - T_0.$$

which must be a solution of the following problem:

$$\hat{c}\hat{\rho}\frac{\partial u}{\partial t} = \hat{\lambda}\frac{\partial^2 u}{\partial x^2}, \qquad (t > 0, \quad -\infty < x < 0)$$

$$u(0, x) = 0$$

$$u(t, 0) = T_{\mathrm{surf}}(t) - T_0$$

$$\left| \lim_{x \to -\infty} u(t, x) \right| < \infty$$

Heat flux in a blast furnace wall

The Laplace transform gives:

$$s\,\hat{c}\hat{\rho}U(s, x) = \hat{\lambda}\frac{d^2 U(s, x)}{dx^2}$$

The solution bounded at $x \to -\infty$ is:

$$U(s, x) = U(s, 0)\exp\left(x\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\right)$$

Then we have

$$\frac{dU}{dx}(s, x) = U(s, 0)\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\,\exp\left(x\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\right)$$

Heat flux in a blast furnace wall

$$U(s, x) = U(s, 0)\exp\left(x\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\right) \qquad \frac{dU}{dx}(s, x) = U(s, 0)\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\,\exp\left(x\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\right)$$

$$\frac{1}{\sqrt{s}}\frac{dU}{dx}(s, 0) = \sqrt{\frac{\hat{c}\hat{\rho}}{\hat{\lambda}}}\,U(s, 0)$$

The inverse Laplace transform gives:

$${}_0D_t^{-1/2}\frac{\partial u}{\partial x}(t, 0) = \sqrt{\frac{\hat{c}\hat{\rho}}{\hat{\lambda}}}\,u(t, 0)$$

After fractional integration of both sides:

$$\frac{\partial u}{\partial x}(t, 0) = \sqrt{\frac{\hat{c}\hat{\rho}}{\hat{\lambda}}}\,{}_0D_t^{1/2}u(t, 0)$$

Heat flux in a blast furnace wall

$$\frac{\partial u}{\partial x}(t, 0) = \sqrt{\frac{\hat{c}\hat{\rho}}{\hat{\lambda}}}\,{}_0D_t^{1/2}u(t, 0) \qquad u(t, x) = T(t, x) - T_0,$$

$$q_A(t) = \sqrt{\hat{c}\hat{\rho}\hat{\lambda}}\,{}_0D_t^{1/2}g(t), \quad g(t) = T_{\mathrm{surf}}(t) - T_0$$

$$q_A(t) = \hat{\lambda}\frac{\partial T}{\partial x}(t, 0)$$

the heat flux

Heat flux in a blast furnace wall

Numerical solution: short memory principle

$$q_A(t) \approx \tilde{q}_A(t) = \sqrt{\hat{c}\hat{\rho}\hat{\lambda}}\ {}_{(t-L)}D_t^{1/2} g(t)$$

Normalized error:

$$\delta_0 = \frac{|q_A(t) - \tilde{q}_A(t)|}{M} = \frac{1}{\sqrt{L}\,\Gamma(\frac{1}{2})}\ , \quad M = \max_{[0,\,\infty]} |g(t)|$$

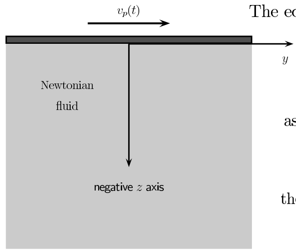
Condition on the memory length: $L \geq \frac{1}{\pi \delta_0^2}$

Calculate using:

$${}_{(t-L)}D_t^{1/2} g(t) = \tau^{-\alpha} \sum_{i=0}^{N(t)} c_i g(t - i\tau)\ ,$$

$$N(t) = \min\left\{ \left[\frac{t}{\tau}\right],\ \left[\frac{L}{\tau}\right] \right\}\ , \quad c_i = (-1)^i \binom{1/2}{i}$$

Bagley-Torvik equation



The equation of motion of the fluid is

$$\rho \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial z^2}, \qquad (0 < t < \infty, \quad -\infty < z < 0)$$

assume that initially the fluid is in equilibrium state

$$v(0, z) = 0, \qquad (-\infty < z < 0)$$

the influence of the plate's motion vanishes for $z \to \infty$:

$$v(t, -\infty) = 0, \qquad (0 < t < \infty)$$

The fluid's velocity at $z = 0$ is equal to the given velocity of the plate:

$$v(t, 0) = v_p(t)$$

Bagley-Torvik equation

Applying the Laplace transform gives:

$$\rho s V(s, z) = \mu \frac{d^2 V(s,z)}{dz^2}$$
$$V(s, 0) = V_p(s),$$
$$V(s, -\infty) = 0,$$

The solution of this problem is:

$$V(s, z) = V_p(s) \exp\left(z \sqrt{\frac{\rho s}{\mu}}\right)$$

Bagley-Torvik equation

By differentiation we obtain:

$$\frac{dV(s,z)}{dz} = \sqrt{\frac{\rho s}{\mu}} V_p(s) \exp\left(z \sqrt{\frac{\rho s}{\mu}}\right) = \sqrt{\frac{\rho s}{\mu}} V(s, z)$$

Knowing the velocity, we can obtain the shear stress:

$$\sigma(t, z) = \mu \frac{\partial v(t, z)}{\partial z}$$

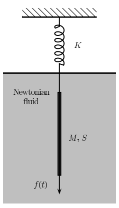
which in terms of the Laplace transforms reads:

$$\overline{\sigma}(s, z) = \mu \frac{dV(s, z)}{dz} = \sqrt{\mu \rho s}\, V(s, z).$$

and therefore, in time domain we have:

$$\sigma(t, z) = \sqrt{\mu \rho}\, {}_0D_t^{1/2} v(t, z)$$

Bagley-Torvik equation



Considering the forces we have:

$$My''(t) = f(t) - Ky(t) - 2S\sigma(t, 0)$$

Using the relationships

$$\sigma(t, z) = \sqrt{\mu\rho}\, {}_0D_t^{1/2} v(s, z) \qquad v_p(t, 0) = y'(t)$$

we arrive at the following equation:

$$Ay''(t) + B\, {}_0D_t^{3/2} y(t) + Cy(t) = f(t) \qquad (t > 0),$$

$$A = M, \quad B = 2S\sqrt{\mu\rho}, \quad C = K,$$

add initial conditions! $\quad y(0) = 0, \qquad y'(0) = 0$

Short memory principle for FDEs

Consider the problem:

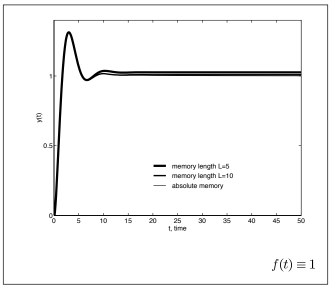
$${}_0D_t^{3/2} y(t) + y(t) = f(t), \qquad (t > 0)$$

$$y(0) = y'(0) = 0$$

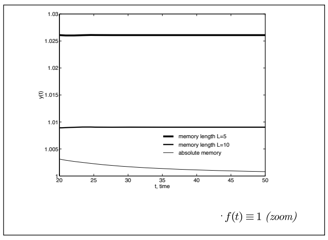
for the following RHS:

1. $f(t) \equiv 1$
2. $f(t) = te^{-t}$
3. $f(t) = t^{-1} e^{-1/t}$
4. $f(t) = e^{-t} \sin(0.2t)$

Short memory principle for FDEs

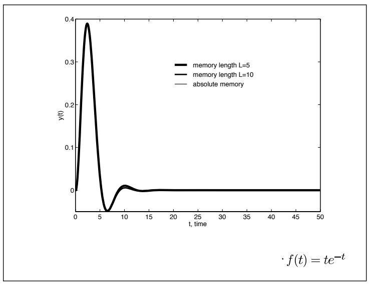


$$ {}_0D_t^{3/2}y(t) + y(t) = f(t), \qquad (t > 0) $$
$$ y(0) = y'(0) = 0 $$

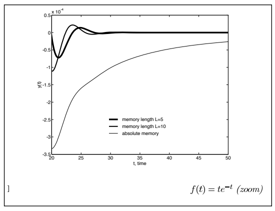


$$ f(t) \equiv 1 $$

Short memory principle for FDEs

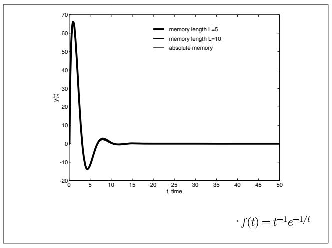


$$ {}_0D_t^{3/2}y(t) + y(t) = f(t), \qquad (t > 0) $$
$$ y(0) = y'(0) = 0 $$

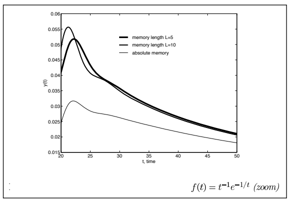


$$ f(t) = te^{-t} $$

Short memory principle for FDEs

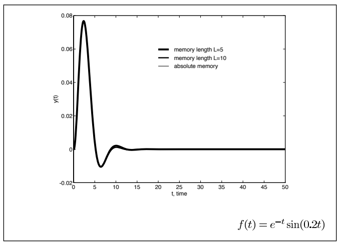


$$ {}_0D_t^{3/2}y(t) + y(t) = f(t), \qquad (t > 0) $$
$$ y(0) = y'(0) = 0 $$

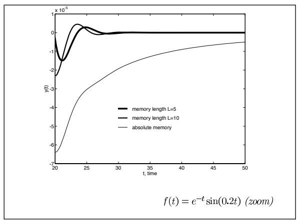


$$ f(t) = t^{-1}e^{-1/t} $$

Short memory principle for FDEs



$$ {}_0D_t^{3/2}y(t) + y(t) = f(t), \qquad (t > 0) $$
$$ y(0) = y'(0) = 0 $$



$$ f(t) = e^{-t}\sin(0.2t) $$